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WEIGHTED SENSITIVITY MINIMIZATION OF LINEAR PERIODIC TIME-VARYING SYSTEMS

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ABSTRACT

The problem of weighted sensitivity minimization of linear periodic time-varying plants is considered. The controllers are also required to be linear periodic with the same period as that of the plant. A formula for the minimal weighted sensitivity is presented.

1. INTRODUCTION

In his pioneering paper [11], Zames formulated the weighted sensitivity optimization problem in the H^∞-norm. Since then a considerable amount of work has been done on various aspects of this problem. We refer the interested reader to the recent survey paper of Francis and Doyle [5] for a good exposition and a complete bibliography. Feintuch and Francis [2,3] showed that H^∞-optimal control problems can be formulated and solved for linear time-varying systems using a resolution space framework. In particular, they showed the existence of an optimal controller and obtained a formula for the optimal cost.

In a recent paper [4], we have considered the weighted sensitivity minimization problem for linear periodic time-varying systems. Here we will present a summary of the results obtained in [4], to which the reader is referred for complete details and proofs. Our main result is a fairly explicit formula for the minimal weighted sensitivity.

2. TRANSFER FUNCTIONS FOR PERIODIC SYSTEMS

Results on transfer function theories for discrete-time linear periodic time-varying systems have been obtained by Davis [1], Jury and Mullin [6], Meyer and Burrus [9], and the references cited there. Here we will present the relevant results of Khargonekar, Poolla, and Tannenbaum [8], who established the fundamental properties of periodic operators following certain ideas from the book [10] of Sz.Nagy and Foias.

For $n>0$, let Ω^n denote the vector space of all sequences $\{x(k)$ in R^n: $k>0\}$. Let $R^n[[z^{-1}]]$ denote the vector space of all formal power series in z^{-1} with coefficients in R^n. As is well known, Ω^n and $R^n[[z^{-1}]]$ are algebraically isomorphic via the isomorphism (z-transform)

$$\tau:\ \Omega^n \to R^n[[z^{-1}]]:\ x \to \sum_{k=0}^{\infty} x(k)z^{-k} .$$

Let Λ: $\Omega^n \to \Omega^n$ denote the right shift operator, i.e., $(\Lambda x)(j) = x(j-1)$ for $j>0$ and $(\Lambda x)(0) = 0$. For any n, $N>0$, define the isomorphism

$$W:\ \Omega^n \to (\Omega^n)^N:\ x \to (x_1,x_2,\dots x_N)'$$

where $x_i(j) = x(i-1+Nj)$. An easy but very important fact is that

$$W\Lambda^N = \Lambda W .$$

Let f: $\Omega^m \to \Omega^p$ be an m-input, p-output, linear time-varying, causal, input-output map. The map f is called N-periodic if and only if

$$f\Lambda^N = \Lambda^N f ,$$

i.e., f commutates with the N-th power of the shift operator. The input-output map induces via W the map

$$\tilde{f}:\ (\Omega^m)^N \to (\Omega^p)^N:\ \tilde{f} = WfW^{-1} .$$

Now if f is N-periodic then it is easy to see that $\tilde{f}\Lambda = \Lambda\tilde{f}$, i.e., $\tilde{f}$ commutes with the shift operator and hence defines a <u>linear time-invariant system</u>. It follows that we <u>can associate a (unique) transfer</u> function $T_f(z)$ with the linear time-invariant map $\tilde{f}$. Indeed, $T_f(z)$ is uniquely defined by

(1) $\tau(\tilde{f}(u)) = T_f(z)\tau(u)$, for all u in $(\Omega^m)^N$

We can also write $T_f(z)$ in the formal power series form

$$T_f(z) = \sum_{i=0}^{\infty} T_i z^{-i} ,$$

Now f is causal if and only if T_0 is <u>lower triangular</u>. We thus see that to any linear <u>periodic time-varying</u> causal input-output map we can associate a unique transfer matrix $T_f(z)$ such that $T_f(\infty)$ is lower triangular. Conversely, given any pNxmN proper transfer matrix $T(z)$ such that $T(\infty)$ is lower triangular, we can define a linear periodic time-varying m-input, p-output map f such that $T(z) = T_f(z)$.

This one-to-one correspondence between periodic systems and the "larger" time-invariant systems is natural from both system theoretic and mathematical points of view. All the standard formulae for interconnection of systems hold. Also f has a finite-dimensional realization if and only if $T_f(z)$ is a rational matrix. A periodic input-output map f is called <u>stable</u> if

$$f:\ (h^2)^m \to (h^2)^p$$

is a bounded linear operator. (Here $h^2(\quad\Omega)$ is the usual Hilbert space of square summable sequences.) In this case we define

$$\|f\| = \sup\{\|f(u)\|:\ u \text{ in } (h^2)^m,\ \|u\| < 1\}.$$

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It can be shown [8] that

(2) $\|f\| = \|T_f\|_\infty := \sup\{\sigma(T_f(e^{j\omega})): \omega \text{ in } [0,2\pi]\}$

where $\sigma(M)$ denotes the largest singular value of M. It can also be shown that b is stable if and only if the entries of T_f belong to the Hardy space H^∞ of essentially bounded analytic functions in the complement of the closed unit disc.

3. WEIGHTED SENSITIVITY MINIMIZATION

Let P be an N-periodic linear time-varying plant. Let W_1, W_2 be stable, causal N-periodic weighting functions such that W_1^{-1} and W_2^{-1} exist and are stable and causal. Following Zames [11], consider the optimization problem:

(3) minimize $\|W_1(I+PC)^{-1}W_2\|$

over all internally stabilizing controllers C. (It is straightforward to give a minimax optimal control interpretation of this problem in the setting of disturbance attenuation and tracking problems.)

Let $G := T_p$ be the associated transfer matrix of the periodic operator P. We will assume that G has left and right coprime factorizations:

(4) $G = NM^{-1} = M_1^{-1}N_1$

with the Bezout identities.

(5) $UM + VN = I, \; M_1U_1 + N_1V_1 = I$

where matrices M, N, M_1, N_1, U, V, U_1, V_1, have entries in H^∞ and are lower triangular at ∞. In other words, they represent stable causal linear periodic time-varying operators. If G is a rational matrix (i.e., P is finite-dimensional) then such factorizations and Bezout identities always exist.

With these coprime factorizations,

(6) $\{K = (V_1 + MQ)(U_1 - NQ)^{-1}$: Q with entries in H^∞ such that $Q(\infty)$ is lower triangular$\}$

is the set of transfer functions of all causal N-periodic internally stabilizing controllers C. Let $\hat{W}_1$ and $\hat{W}_2$ represent transfer functions of the weighting operators W_1 and W_2. Now using the above parameterization of all stabilizing controllers and (2), our optimization problem becomes:

(7) minimize $\|\hat{W}_2(U_1 - NQ)M_1\hat{W}_1\|_\infty$

over all Q with entries in H^∞ such that $Q(\infty)$ is lower triangular. To simplify further, we have the following proposition on inner-outer factorizations.

(8) PROPOSITION. Let A be a rational matrix with entries in H^∞ such that $A(\infty)$ is lower triangular. Then if A has full row (resp., column) rank on the unit circle then we can find rational matrices A_i, A_o (resp., $\hat{A}_o, \hat{A}_i$) with entries in H^∞ such that

$$A = A_iA_o \text{ (resp., } A = \hat{A}_o\hat{A}_i)$$

where A_i (resp., $\hat{A}_i$) is inner, A_o (resp., $\hat{A}_o$) is outer, and $A_i(\infty)$, $A_o(\infty)$ (resp., $\hat{A}_i(\infty)$, $\hat{A}_o(\infty)$ are lower triangular.

In order to simplify (7), let us assume that $\hat{W}_2N$ is full row rank and $M_1\hat{W}_1$ is full column rank on the unit circle. This corresponds to assuming that the plant Q is free of poles and zeros on the unit circle and that the weighting functions are free of zeros on the unit circle. With this assumption, consider the inner-outer factorizations

$$\hat{W}_2N = (\hat{W}_2N)_i(\hat{W}_2N)_o, \quad M_1\hat{W}_1 = (M_1\hat{W}_1)_o(M_1\hat{W}_1)_i \; ,$$

Define

$$T = (\hat{W}_2N)_i{}^* \, \hat{W}_2U_1(M_1\hat{W}_1)_i{}^*,$$

$$V = (\hat{W}_2N)_oQ(M_1\hat{W}_1)_o \; .$$

Then it is easy to see that our optimization problem reduces to

(9) minimize $\|T-V\|_\infty$

over all V with entries in H^∞ such that $V(\infty)$ is lower triangular.

Note that T does not necessarily have its entries in H^∞. However, T does have its entries in L^∞ of the unit circle.

Before stating the main result of this paper, we need some notation on Hankel operators. Given any matrix A with entries in the L^∞ of the unit circle, let

(10) $A = \sum_{i=-\infty}^{\infty} \alpha_i z^{-i}$

be its Fourier series expansion. Let H(A) denote the infinite Hankel matrix

(11) $H(A) = [\alpha_{i+j-1}]_{i,j=1,2,\ldots}$.

Note that H(A) depends only on the positive Fourier coefficients of A.

We can now state the main result:

(12) THEOREM. Consider the optimization problem (9). Then

$$\min\{\|T-V\|_\infty: V \text{ with entries in } H^\infty, V(\infty) \text{ lower triangular}\}$$

$$= \max\{\|H(T_j)\|: j=1,2,\ldots,N\}$$

where

$$T_j = E_jTE_j^{-1}$$

$$E_j := \text{diag}\,(\underbrace{zI, zI, \ldots, zI}_{(j-1)\text{ blocks}}, I, I, \ldots, I)$$

This result shows that the minimal weighted sensitivity can be computed by obtaining the maximum of norms of a finite collection of Hankel operators. If T is a rational matrix, these norms can be very conveniently computed using state-space realizations. For details and relevant references, see [5].

Finally, we note that it follows from [4,8,7] that the minimal weighted sensitivity can not be improved by using arbitrary nonlinear time-varying controllers. In other words, solution to the optimization problem (3) is not lowered by allowing C to be any nonlinear time-varying internally stabilizing controller as compared to restricting C to be a linear periodic time-varying controller.

References

[1] J. H. Davis, "Stability conditions derived from spectral theory: discrete systems with periodic feedback", SIAM J. Control, 10, 1972, pp. 1-13.

[2] A. Feintuch and B. A. Francis, "Uniformly optimal control of linear time-varying systems", Systems and Control Letters, 5: 67-71, 1984.
[3] A. Feintuch and B. A. Francis, "Uniformly optimal control of linear feedback systems", to appear in Automatica.
[4] A. Feintuch, P. P. Khargonekar, and A. Tannenbaum, "On the sensitivity minimization problem for linear time-varying periodic systems", to appear in SIAM J. Control and Optimization.
[5] B. A. Francis and J. C. Doyle, "On an optimal control problem with H^∞-norm criterion", to appear.
[6] E. I. Jury and F. J. Mullin, "The analysis of sampled-data control systems with a periodically time-varying sampling rate", IRE Trans. Auto. Control, AC-24, 1959, pp. 15-21.
[7] P. P. Khargonekar and K. R. Poolla, "Uniformly optimal control of linear time-invariant plants: nonlinear time-varying controllers", submitted for publication.
[8] P. P. Khargonekar, K. R. Poolla, and A. Tannenbaum, "Robust control of linear time-invariant plants by periodic compensation", to appear in the IEEE Trans. on Automatic Control, November 1985.
[9] R. A. Meyer and C. S. Burrus, "A unified analysis of multirate and periodically time-varying digital filters", IEEE Trans. Circuits and Systems, CAS-22, 1975, pp. 162-168.
[10] B. Sz-Nagy and C. Foias, Harmonic Analysis of Operators on Hilbert Space, American Elsevier, NY, 1970.
[11] G. Zames, "Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses", IEEE Trans. Auto. Contr., AC-26, 1981, pp. 301-320.